Lecture 1

DEFINITE INTEGRALS

In calculus we define <u>indefinite</u> integrals

$$\int f(x)\,dx = F(x) + c, \quad c \text{ is an arbitrary constant}$$

$f(x)$ is called <u>the integrand</u>
x is the <u>integration variable</u>
c is called the constant of integration

Function $F(x)$ is called an <u>anti-derivative</u> or primitive function
It is any function such that

$$F'(x) = f(x)$$

Example 1

$$\int x \, dx = \frac{x^2}{2} + C$$

Example 2

$$\int x^2 \, dx = \frac{x^3}{3} + C$$

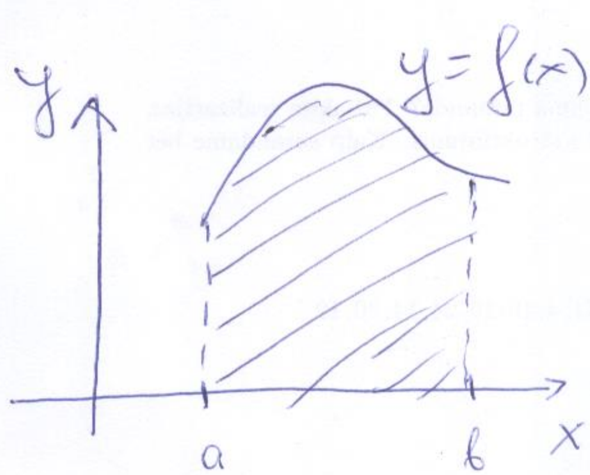
It was explained for you that F exists if the integrand function

$$f \in C[a, b]$$ (a sufficient condition)

How to prove this fact?

We start by solving practical problems:

how ~~to~~ caculated areas, volumes, find displacement from velocity.



S

The area of the region bounded by ~~f(x)~~ the above graph of $f(x)$, base - x axis, and vertical lines at $x=a$ and $x=b$.

Procedure:

- define a partition of a closed interval $[a, b]$

$$a = x_0 < x_1 < x_2 < \dots < x_n = b.$$

The width of subinterval $[x_{i-1}, x_i]$ is denoted as

$$\Delta x_i = x_i - x_{i-1}, \quad i = 1, 2, \dots, n.$$

The space mesh width is defined as

$$\Delta = \max_{1 \le i \le n} \Delta x_i$$

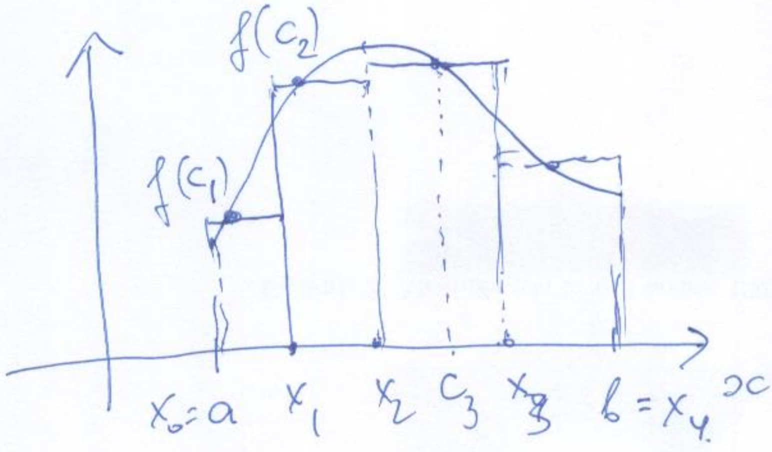
- select in each subinterval $[x_{i-1}, x_i]$ a point

$$c_i \in [x_{i-1}, x_i], \quad i = 1, \dots, n$$

- define rectangles, with base length $\Delta x_i = x_i - x_{i-1}$, and height $f(c_i)$.

The area of such a rectangle is equal to

$$s_i = f(c_i) \cdot \Delta x_i$$



The sum of areas of all rectangles is calculated as

$$\sigma = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

It give approximations of the exact value S.

Taking more points of the mesh and making $\Delta \to 0$ we get in a limit the required value of S:

$$S = \lim_{\Delta \to 0} \sigma = \lim_{\Delta \to 0} \sum_{i=1}^{n(\Delta)} f(c_i)\Delta x_i$$

Definition of definite integrals.

Consider a bounded function $f(x)$ in the closed interval $[a, b]$.
(as an example take continuous functions $f \in C[a,b]$)

1. Define a mesh in $[a, b]$ (a partition of this interval)

$$a = x_0 < x_1 < x_2 < \dots < x_n = b.$$

2. Select a point in each subinterval $[x_{i-1}, x_i]$:

$$c_i \in [x_{i-1}, x_i], \quad i = 1, 2, \dots, n$$

3. Calculate a sum of f over $[a, b]$

$$\sigma_n = \sum_{i=1}^{n} f(c_i) \Delta x_i$$

This sum is called a $\boxed{\text{Riemann sum.}}$

4. Find a limit as $\Delta \to 0$:

$$\lim_{\Delta \to 0} \sum_{i=1}^{n(\Delta)} f(c_i)\,\Delta x_i .$$

If a limit exists and don't depend on selection of points of meshes (partitions) and points c_i, then this limit of discrete sums is called $\boxed{\text{the integral of } f \text{ in the interval } [a,b]}$ and f is called Riemann integrable, notation

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{i=1}^{n(\Delta)} f(c_i)\,\Delta x_i .$$

a is the lower limit and b is the upper limit for a function $f(x)$.

Example 3. Compute

$$\int_0^1 x\,dx.$$

Since a limit of a Riemman sum don't depend on a mesh, we use a uniform mesh

$$x_i = \frac{1}{n} i, \quad i = 0, 1, \dots, n$$

and take

$$c_i = x_i, \quad i = 1, 2, \dots, n.$$

Then we get a sum: $\left(x_i - x_{i-1} = \frac{1}{n} \right)$

$$\sigma_n = \sum_{i=1}^{n} \left(\frac{i}{n}\right) \times \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i$$

Recalled that for the progression sequence

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

thus

$$\sigma_n = \frac{1}{2} + \frac{1}{2n} \xrightarrow[n \to \infty]{} \frac{1}{2} = \int_0^1 x\,dx.$$

Example 4 Calculate:

$$\int_0^1 x^2 dx.$$

Hint: $1^2 + 2^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$

(mathematical induction can be used ~~to~~ to get a proof of it).

Remark 1. The integral $\int_a^b f(x)\,dx$ depends not only on the integrand function f, but also on the interval $[a,b]$.

The same function f can be integrable in one interval, but not integrable in the other interval.

<u>Remark 2</u>. In order to define sums σ_n we require that f be a <u>bounded function</u>.

For non-bounded functions integrals also can be defined, but we must relax (change) <u>the definition</u>. Here we consider <u>only Riemann definite integrals</u>.

Next we give some ~~p~~ strict justification when a limit of integrable sums exist.

It is clear that we restrict ~~of~~ a class of functions f, suited to guarantee that the proof is valid.

Darboux sums

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{(the supremum)}$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \quad \text{(the infimum)}$$

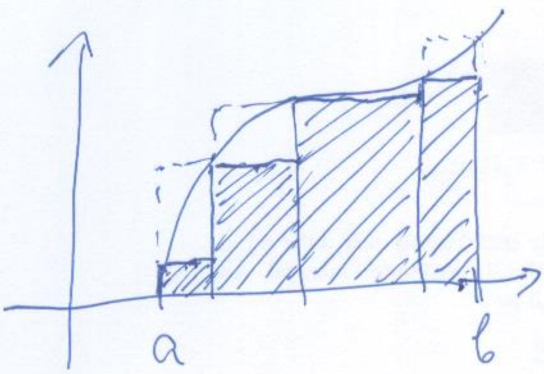
The <u>upper Darboux</u> sum of f with respect to mesh $P = (x_0, \dots, x_n)$ is

$$U_n = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$$

and the <u>lower Darboux</u> sum of f with respect to P is

$$L_n = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i .$$

Geometrical view:



The lower and upper sums.

Remark. For a given mesh P:

$$L_n \leq \sum_{i=1}^{n} f(c_i)\,\Delta x_i \leq U_n.$$

Lemma 1. If a new point is added to P, then lower sum can only increase and upper sum to decrease.

$$L_n \leq L_{n+1} \leq U_{n+1} \leq U_n.$$

Proof. Let's assume that $\tilde{x}$ is added to $[x_{i-1}, x_i]$:

$$x_{i-1} < \tilde{x} < x_i.$$

We define

$$m_i = \inf_{x_{i-1} \le x \le x_i} f(x),$$

$$\omega_i = \inf_{x_{i-1} \le x \le \tilde{x}} f(x),$$

$$\gamma_i = \inf_{\tilde{x} \le x \le x_i} f(x).$$

~~A~~ A difference between new lower sum and L is equal to

$$(\omega_i - m_i)(\tilde{x} - x_{i-1}) + (\gamma_i - m_i)(x_i - \tilde{x})$$

Since

$$\omega_i \geq m_i, \quad \gamma_i \geq m_i,$$

then $\underline{L_n \le L_{n+1}}$. The proof of the second estimate is similar.

<u>Lemma 2</u>. Let consider two different partitions P_1 and P_2 of $[a, b]$. Then

$$L_{P_1} \le U_{P_2}.$$

Proof. Let denote lower and upper sums corresponding ~~to~~ P_1 and P_2 as

$$L_{P_1},\ L_{P_2},\ U_{P_1} \text{ and } U_{P_2}.$$

Next we consider one more mesh $P_3 = P_1 \cup P_2$. It follows from Lemma 1 that

$$\underbrace{L_{P_1} \le L_{P_3}}_{\text{lemma 1}} \overbrace{\le}^{\text{definition}} \underbrace{U_{P_3} \le U_{P_2}}_{\text{lemma 1.}}$$

Lemma 3.
Since a set of $\{L_n\}$ is bounded from above by any U_m, then $\exists : \underline{I}$

$$\underline{I} = \sup_{p} L_n$$

(the exact bound from above
exact upper bound)

Similarly

$$\overline{I} = \inf_{p} U_n$$

(the exact bound from below, exact lower bound)

$$\boxed{L \le \underline{I} \le \overline{I} \le U}$$

Theorem 1. A function f is Riemann integrable in $[a, b]$ if and only if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for every mesh P with $\Delta \leqslant \delta$ the estimate

$$U_P - L_P < \varepsilon \quad (\text{i.e. } \lim_{\Delta \to 0} (U - L) = 0)$$

Proof. Necessity. Let's assume that function f integrable in $[a, b]$, i.e.

$$\lim_{\Delta \to 0} \sum_{i=1}^{n(\Delta)} f(c_i) \Delta x_i = I \quad \left(\begin{matrix}\text{limit} \\ \text{exists}\end{matrix}\right).$$

It mean that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \text{ for } \Delta < \delta(\varepsilon) \Rightarrow$$

$$|\sigma_n - I| < \frac{\varepsilon}{2}.$$

Then it follows that

$$I - \frac{\varepsilon}{2} < \sigma_n < I + \frac{\varepsilon}{2}.$$

Next we fix the partition P with $\Delta < \delta$:

$$P = (x_0, x_1, \dots, x_n).$$

Then the upper and lower sums are also fixed, but σ_n can change for different sets of c_i, $i = 1, \dots, n$.

It is clear that U_p and L_p are ~~a~~ exact upper and lower bounds of the set $\{\sigma_n\}$ (for different c_i)

$$I - \frac{\varepsilon}{2} < L_p \le \sigma_n \le U_p < I + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad U_p < I + \frac{\varepsilon}{2}$$

$$L_p > I - \frac{\varepsilon}{2} \Rightarrow (-L_p) < \left(\frac{\varepsilon}{2} - I\right)$$

$$\Rightarrow \quad \boxed{U_p - L_p < \varepsilon}$$

2. Sufficiency

Let assume, that $\forall \varepsilon > 0$ for some P partition

$$U_P - S_P < \varepsilon \quad (*)$$

It follows from Lemma 3, that

$$\underline{I} = \overline{I} = I.$$

and we have for ~~∀~~ any L, U (from sets of lower and upper sums)

$$L \le I \le U$$

By definition, for any σ_n:

$$L \le \sigma_n \le U$$

Since both I and σ_n ~~so~~ are between L and U and estimate $(*)$ is valid, ~~then~~ for Δ sufficiently small

$$|I - \sigma_n| < \varepsilon \Rightarrow \boxed{\lim_{\Delta \to 0} \sigma_n = I}$$

Let us denote

$$\omega_i = M_i - m_i \quad , \quad i = 1, \dots, n.$$

Then

$$U - L = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

$$= \sum_{i=1}^{n} \omega_i \Delta x_i$$

Then the integrability condition can be rewritten as

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \Delta < \delta$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} \omega_i \Delta x_i < \varepsilon}$$

We use the result, that condition

$$\underline{I} = \overline{I} = I$$

is not only sufficient but also necessary condition for the integrability of f in $[a, b]$.